Device-independent lower bounds on the conditional von Neumann entropy

Peter Brown, Hamza Fawzi and Omar Fawzi

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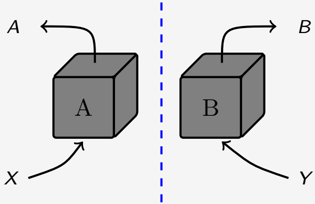
Motivation I

Bell-nonlocality

Observe $P_{AB|XY}$



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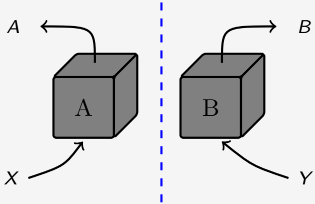

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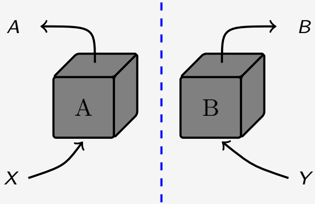

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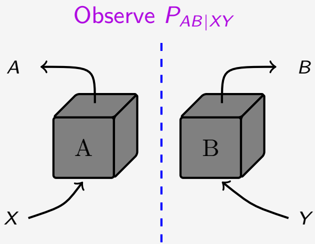

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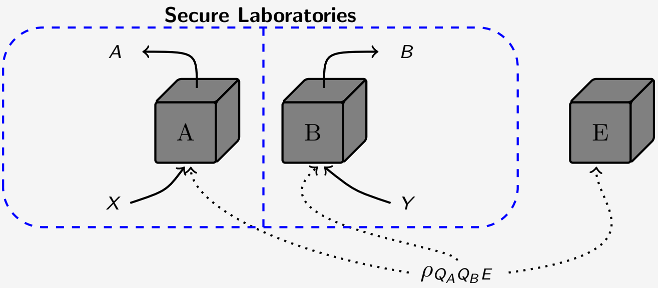
- Foundation for randomness expansion / key-distribution protocols!
- Security and analysis relies on being able to calculate the *rates* of such protocols (bits per round).

Randomness generated per round



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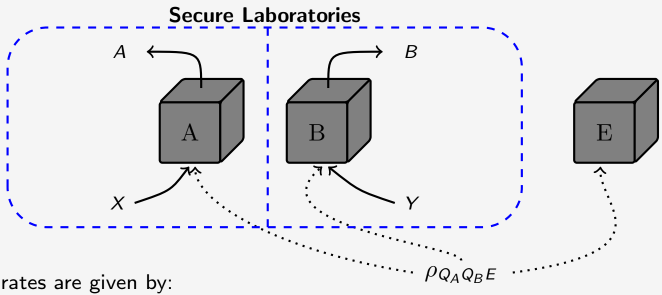

Asymptotic rates are given by:

- **Randomness expansion**

$$H(AB|X = x^*, Y = y^*, E)$$

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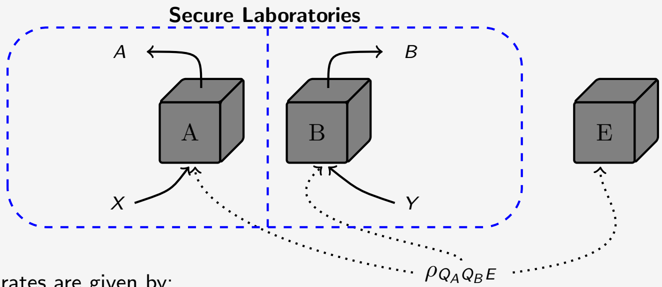

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Previous works

Known approaches

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Our approach

Define a sequence

$$H_m(\rho) = \inf_{Z_1,\ldots,Z_m \in B(H)} \mathrm{Tr}\left[\rho\, q(Z_1,\ldots,Z_m)\right] \tag{1}$$

such that $H_m \leq H$ and $H_m \to H$ as $m \to \infty$.

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The goal

Derive something of the form

$$D(\rho\|\sigma) \leq \sum_{i=1}^{m} \inf_{Z} \mathrm{Tr}\left[\rho p_i(Z)\right] + \mathrm{Tr}\left[\sigma q_i(Z)\right]$$

with p_i and q_i some polynomials and with the RHS converging as $m \to \infty$.

Derivation overview

1. Gauss-Radau approximation of the logarithm

$$\ln(x) \geq \sum_{i=1}^{m} w_i f_{t_i}(x)$$

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3. Each $D_{f_t}(\rho\|\sigma)$ admits a variational form

$$D_{f_t}(\rho\|\sigma) = \frac{1}{t} \inf_{Z \in B(H)} \{\mathrm{Tr}\left[\rho(I + Z + Z^* + (1-t)Z^*Z)\right] + t\mathrm{Tr}\left[\sigma Z Z^*\right]\}$$

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and RHS converges as $m \to \infty$.

Lower bound on $H(A|X = x^*, Q_E)$

$H(A|B) = -D(\rho_{AB} \| I_A \otimes \rho_B)$

Theorem

The rate $\inf H(A|X = x^*, Q_E)$ *is never larger than*

$$c_m + \inf_{strategies} \sum_{i=1}^{m-1} \frac{w_i}{t_i \ln 2} \sum_a \mathrm{Tr}\left[\rho_{Q_A Q_E}(M_{a|x^*} \otimes (Z_{a,i} + Z_{a,i}^* + (1 - t_i) Z_{a,i}^* Z_{a,i}) + t_i Z_{a,i} Z_{a,i}^*)\right] .$$

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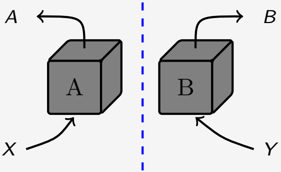
Results

- Applied our method to compute rates for DIRNG and DIQKD.



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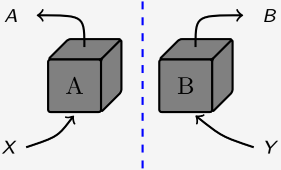

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- Looked at different constraint sets C:
  - CHSH score

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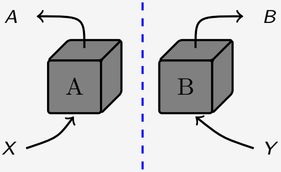

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- Investigated *detection efficiency* noise model.
  - Independent probability $\eta \in [0,1]$ that each device *succeeds*.
  - Device failures recorded as a particular outcome.

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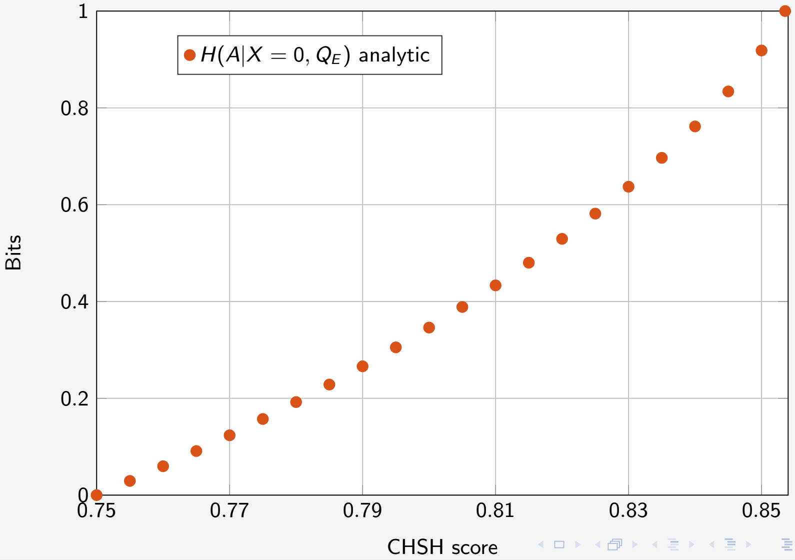
Bounding $\inf H(A|X=0, Q_E)$



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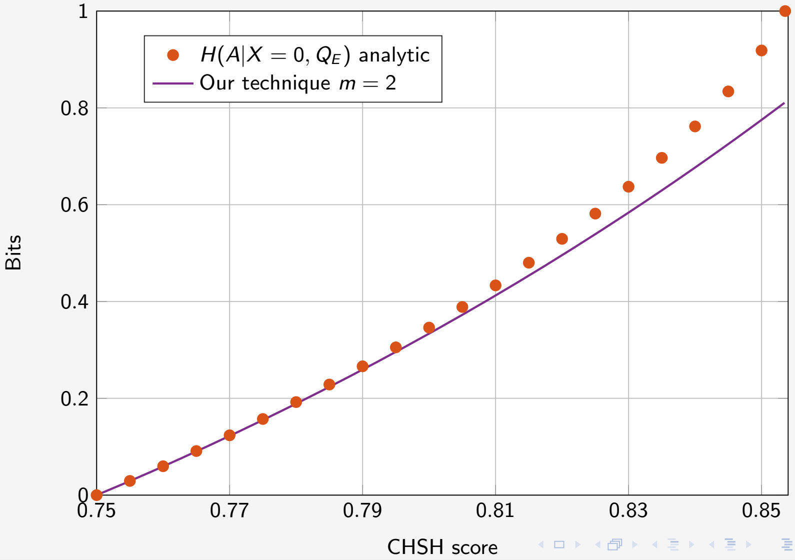

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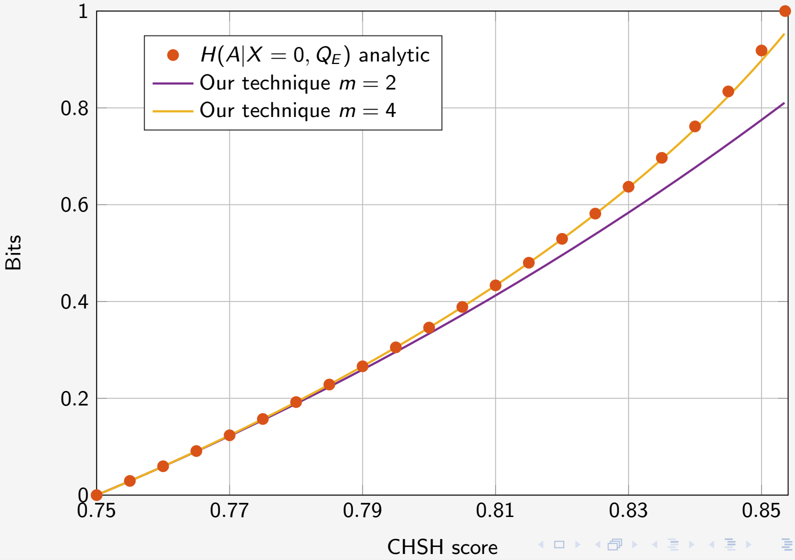

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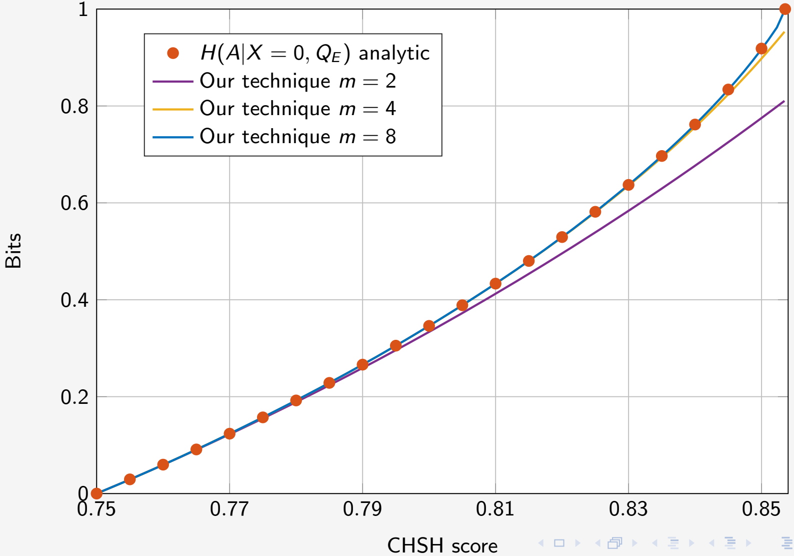
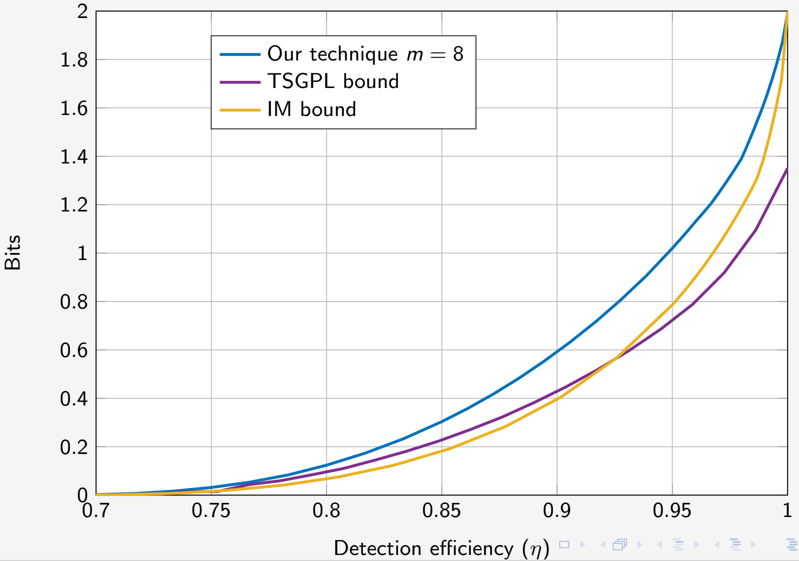


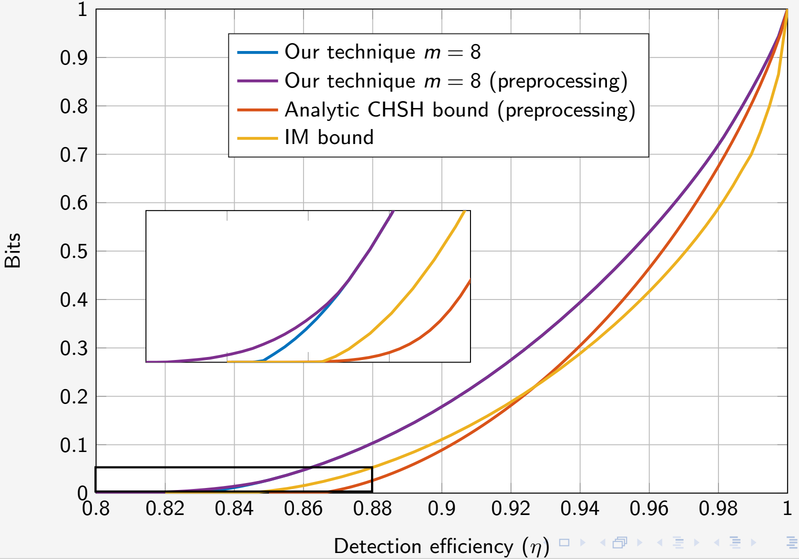
Results II – Improved randomness expansion rates

Bounding $\inf H(AB|X = 0, Y = 0, Q_E)$



Results III – Improved DIQKD rates

Bounding $\inf H(A|X=0, Q_E) - H(A|X=0, Y=2, B)$



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- Is DIQKD feasible now?
- Other entropic quantities?

Bibliography

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